· Finding limits in polar coordinates

· $\lim_{x\to 0}\lim_{y\to 0}$, $\lim_{y\to 0}\lim_{x\to 0}$

· continuity ($\varepsilon - \delta$)

· partial derivatives f_x , f_y (regardly y-variable as constant)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\frac{\partial f}{\partial x}$ $\quad$ $\frac{\partial f}{\partial y}$

Higher order partial derivatives

eg Consider $f(x,y)$

1st order derivate $\frac{\partial f}{\partial x} = f_x$ $\quad\quad$ $\frac{\partial f}{\partial y} = f_y$

2nd order derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\underbrace{\partial y \partial x}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = \underbrace{f_{xy}}$$

order : y then x $\quad\quad$ x then y

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

3rd order derivative $\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right)$

$= ((f_y)_y)_x$

$= f_{yyx}$

$\frac{\partial^3 f}{\partial x^3} = f_{xxx}$

others: f_{xxy}, f_{yxy}, f_{yyy}

f_{xxy}, f_{xyx}, f_{yxx}

[8 possibilities]

Similar for higher order, more variables.

eg $f(x,y) = x\sin y + y^2 e^{2x}$

1st order derivative:

$f_x = \sin y + 2y^2 e^{2x}$, $f_y = x\cos y + 2y e^{2x}$

2nd order derivative

$f_{xx} = (f_x)_x = 4y^2 e^{2x}$, $f_{xy} = \cos y + 4y e^{2x}$

$f_{yx} = \cos y + 4y e^{2x}$, $f_{yy} = -x\sin y + 2e^{2x}$

same

Q Is it always true that $f_{xy} = f_{yx}$?

A No.

eg $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

$f_{xy}(0,0)$ & $f_{yx}(0,0)$?

(sol) $$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

we need to find $f_x(0,h)$ $(h \neq 0)$ and $f_x(0,0)$.

$f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ near $(0,h)$ $(h \neq 0)$.

$$f_x = \frac{(x^2+y^2)(3x^2y - y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$\therefore f_x(0,h)$ for $h \neq 0$ $$= \frac{h^2 \cdot (-h^3) - 0}{h^4}$$

$$= -h$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0 - 0}{h}$$

$$= 0$$

$$\therefore\ f_{xy}(0,0) = \lim_{h\to 0} \frac{-h - 0}{h} = -1.$$

$f_{yx}(0,0)$?

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

Similarly, $f_{yx}(0,0) = 1.$

In summary, $f_{xy}(0,0) = -1$

$f_{yx}(0,0) = 1$

They are different.

(Remark observe that $f(x,y) = -f(y,x)$

Hence $f_{yx}(0,0) = -f_{xy}(0,0) = 1$)

Q When $f_{xy} = f_{yx}$?

Thm (Clairaut's theorem / mixed derivative theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be an open set,

$f : \Omega \to \mathbb{R}$.

If f_{xy}, f_{yx} exist and are continuous on Ω, then $f_{xy} = f_{yx}$ on Ω.

Stronger version:

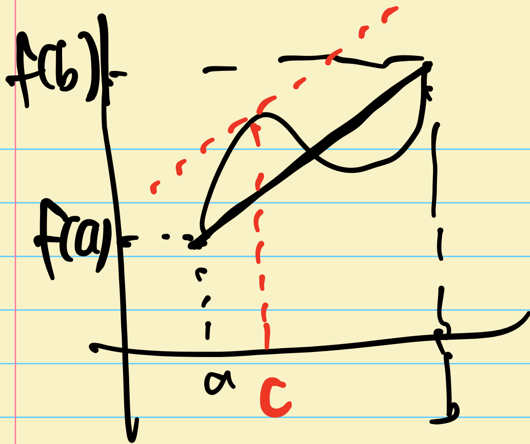
If f_{xy}, f_{yx} exist in an open ball containing $\vec{a}$ and f_{xy} & f_{yx} are continuous at $\vec{a}$, then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$.

Tool for proof)

Mean value theorem for 1-variable function. (MVT)

Recall $f : [a,b] \to \mathbb{R}$, continuous on $[a,b]$ differentiable on (a,b)

Then $\exists c \in (a,b)$ s.t. $\dfrac{f(b) - f(a)}{b - a} = f'(c)$

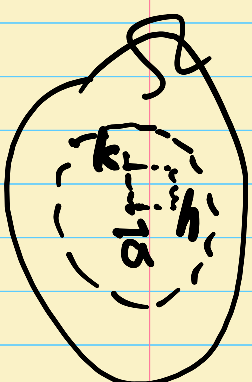


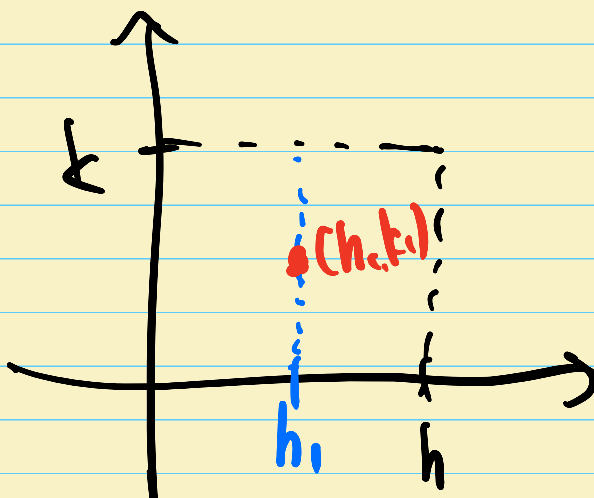
(pf of Clairaut's thm)

We may assume $\vec{a} = (0,0) \in \Omega$

We want to show $f_{xy}(0,0) = f_{yx}(0,0)$.

Let $h, k > 0$ and $[0,h] \times [0,k] \subseteq \Omega$





$$\alpha = f(h,k) - f(0,k) - f(h,0) + f(0,0)$$

Let $g(x) = f(x,k) - f(x,0)$, $\quad 0 \le x \le h$

Then $\alpha = g(h) - g(0)$

$$g'(x) = f_x(x,k) - f_x(x,0)$$

MVT $\Rightarrow \exists h_1 \in (0, h)$

s.t. $\dfrac{g(h)-g(0)}{h} = g'(h_1)$

i.e. $\dfrac{\alpha}{h} = f_x(h_1, k) - f_x(h_1, 0)$

MVT again $\Rightarrow$ to $f_x(h_1, y)$ $\exists k_1 \in (0, k)$ s.t.

$$\frac{f_x(h_1, k) - f_x(h_1, 0)}{k} = f_{xy}(h_1, k_1)$$

$\therefore \alpha = h(f_x(h_1, k) - f_x(h_1, 0))$

$= hk f_{xy}(h_1, k_1)$

If we consider $h(y) = f(h, y) - f(h, 0)$

Similarly, we conclude that

$\exists (h_2, k_2) \in (0, h) \times (0, k)$ s.t.

$\alpha = hk f_{yx}(h_2, k_2)$

$\therefore \alpha = hk f_{xy}(h_1, k_1) = hk f_{yx}(h_2, k_2)$

Take $h, k, \to 0+$.

then $(h_1, k_1), (h_2, k_2) \to (0,0)$

+ continuity of f_{xy} and f_{yx} at $(0,0)$

$$\Rightarrow f_{xy}(0,0) = f_{yx}(0,0) \qquad \square$$

Def Let $\Omega \subseteq \mathbb{R}^n$ be open. $f: \Omega \to \mathbb{R}$.

$r \geq 0$

f is called a <u>C^r-function</u> if all partial derivatives of f up to order r exist and continuous on Ω.

f is called a <u>C^∞-function</u> if it is C^r-function for all $r \geq 0$.

eg. ① f is C^0(function) if it is continuous.

② $f(x,y)$ is C^2 if

$f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist and are continuous.

Examples of C^∞-functions

- polynomials
- rational functions
- exp, log, trigonometric functions
- their sum / difference / product / quotient / compositions

eg $e^{x^2-y} \cdot \sin \frac{x}{y}$

Generalization of Clairaut's thm

Thm If f is C^r on an open set $\Omega \subseteq \mathbb{R}^n$, then the order of differentiation does not matter for all partial derivatives up to r.

eg
- $r=2$; original Clairaut's thm $\Rightarrow$ $f_{xy} = f_{yx}$
- If $f(x,y,z)$ is C^3, then

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}$$

$$f_{xyz} = f_{xzy} = f_{zxy} = f_{zyx} = f_{yxz} = f_{yzx}$$

$$f_{xxy} = f_{xyx} = f_{yxx}. \quad \cdots,$$

Differentiability.

Differentiability in 1-variable:

$f: \mathbb{R} \to \mathbb{R}$ is differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \quad \text{exists.}$$

multi-variable: $f: \mathbb{R}^n \to \mathbb{R}$, $\vec{a} \in \mathbb{R}^n$.

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a})}{\vec{x} - \vec{a}} \quad \text{exist?}$$

→ a vector

does not make sense to divide by a vector.

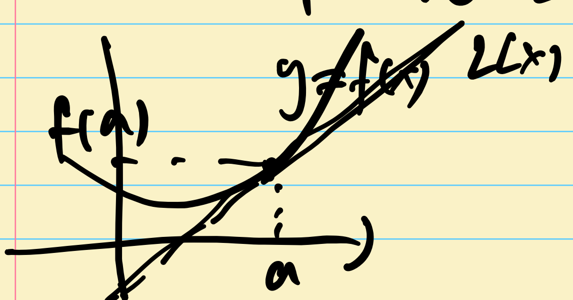
Interpret differentiability as a possibility of linear approximation and an error.

∴ generalize to multi-variable case.

linear approximation of f(x).

Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $\vec{a}$.

Then $f(x) \approx L(x) := f(a) + f'(a)(x - a)$



L(x) is "the best" linear function to approximate f(x) near a.

Error of an approximation $\varepsilon(x) := f(x) - L(x)$.

$$= f(x) - f(a) - f'(a)(x-a)$$

Note that $\dfrac{\varepsilon(x)}{x-a} = \dfrac{f(x) - f(a)}{x-a} - f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a) \iff \lim_{x \to a} \frac{\varepsilon(x)}{x-a} = 0$$

(error is small compared to $x-a$)

$$\left(\iff \lim_{x \to a} \frac{|\varepsilon(x)|}{|x-a|} = 0 \right)$$

Consider a function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

A possible formulation for the differentiability of f at (a, b) is:

$\exists$ a plane $L(x, y) = f(a, b) + C(x-a) + D(y-b)$

which approximates $f(x, y)$ near $(x, y) = (a, b)$

in the sense that $\displaystyle\lim_{(x,y) \to (a,b)} \frac{|f(x,y) - L(x,y)|}{\|(x,y) - (a,b)\|} = 0$

Suppose such limit exists and is 0.

We know that limit along any path should equal.

In particular, along $y=b$, $x \to a^+$

$$0 = \lim_{\substack{(x,y)\to(a,b) \\ y=b \\ x\to a^+}} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|}$$

$$= \lim_{x\to a^+} \frac{f(x,b) - L(x,b)}{\|(x,b) - (a,b)\|}$$

$$= \lim_{x\to a^+} \frac{f(x,b) - L(x,b)}{|x-a|}$$

$$= \lim_{x\to a^+} \frac{f(x,b) - L(x,b)}{x-a}$$

$$= \lim_{x\to a^+} \frac{f(x,b) - f(a,b) - C(x-a)}{x-a}$$

$$= \lim_{x\to a^+} \frac{f(x,b) - f(a,b)}{x-a} - C$$

$$\therefore \quad C = \lim_{x\to a^+} \frac{f(x,b) - f(a,b)}{x-a}$$

Similar, $C = \lim_{x \to a^-} \frac{f(x,b) - f(a,b)}{x-a}$.

$\therefore$ For the plane L "to have a chance" to approximate f near (a,b), the partial derivative $f_x(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}$

must exist and $C = f_x(a,b)$.

Similarly, $f_y(a,b)$ must exist and $D = f_y(a,b)$

In this case, L must be

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$(f_x(a,b), f_y(a,b)$ exist$)$

※: This is necessary condition not sufficient for f to be approximated by a plane near (a,b)

Def $\Omega \subseteq \mathbb{R}^n$ open, $\vec{a} = (a_1 \cdots a_n) \in \Omega$

$f : \Omega \to \mathbb{R}$.

f is <u>differentiable at $\vec{a}$</u> if

- Each $\frac{\partial f}{\partial x_i}(\vec{a}) = f_{x_i}(\vec{a})$ exists for all $i = 1, \cdots, n$.
- For $L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)$

and $\varepsilon(\vec{x}) = f(\vec{x}) - L(\vec{x})$,

$$\lim_{\vec{x} \to \vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0.$$

Rmk

- $L(\vec{x})$ is a linear function
- $L(\vec{a}) = f(\vec{a})$
- $\frac{\partial L}{\partial x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$
- The graph of $L(\vec{x})$ is the n-dimensional "tangent plane" to the graph of $f(\vec{x})$ at $(\vec{a}, f(\vec{a}))$.

Example $f(x,y) = x^2 y$

f is differentiable at $(1,2)$.

(sol) $f_x = 2xy$, $f_y = x^2$

$f_x(1,2) = 4$, $f_y(1,2) = 1$

linear approximation:

$$L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$= 2 + 4(x-1) + (y-2)$$

error $\varepsilon(x,y)$

$$= f(x,y) - L(x,y)$$

$$= x^2y - 2 - 4(x-1) - (y-2)$$

we need to show that

$$\lim_{(x,y)\to(1,2)} \frac{x^2y - 2 - 4(x-1) - (y-2)}{\|(x,y) - (1,2)\|} = 0$$

$$= \lim_{(x,y)\to(1,2)} \frac{x^2y-2-4(x-1)-(y-2)}{\sqrt{(x-1)^2+(y-2)^2}}$$

Let
$h = x-1$
$k = y-2$

$$= \lim_{(h,k)\to(0,0)} \frac{(1+h)^2(2+k)-2-4h-k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2k+2hk+2h^2}{\sqrt{h^2+k^2}}$$

Let
$h = r\cos\theta$
$k = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta+2r^2\cos\theta\sin\theta+2r^2\sin^2\theta}{r}$$

$$= \lim_{r\to 0} r^2\cos^2\theta\sin\theta+2r\cos\theta\sin\theta+2r\sin^2\theta$$

$$= 0$$